LOGIC AND PROOF FOR MATHEMATICIANS
A TWENTIETH CENTURY PERSPECTIVE

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INTRODUCTION AND BACKGROUND

If there is one aspect of mathematics education that frustrates both students and teachers alike, it has got to be learning how to do valid proofs. Students often feel they really know the mathematics they're studying but that their teachers place some unreasonably stringent demands upon their arguments. Teachers, on the other hand, can't understand where their students get some of the wacky arguments they come up with. They argue in circles, they end up proving a different result from what they claim, they make false statements, they draw invalid inferences - it can be quite exasperating at times! Unfortunately, constructing a genuine demonstration is not a side issue in mathematics; deduction forms the backbone and hygiene of mathematical intercourse and simply must be learned by mathematics majors.

I'd like to focus attention on this topic in this paper first by sharing with you how we attack the problem at Dordt. I will then attempt to give the issue a bit of historical and philosophical depth by tracing some developments in mathematics and logic which seem to have had a strong influence upon contemporary approaches to teaching proof. I hope this will contribute to further discussion of the problem. The issue is a perennial one and should be faced squarely by mathematics educators.

At Dordt I teach a one semester, sophomore level course for mathematics majors called MATHEMATICAL PROOF AND THE THEORY OF CALCULUS. At the time when students take this course they have already had two semesters of calculus, which they took along with a number of other majors. Dordt doesn't have enough mathematics majors to justify teaching a section of calculus just for them, so they take a course which could be characterized as calculus for scientists and engineers. As you all know, such a course is largely algorithmic or cookbok-ish in nature; proof is largely absent - though the engineers in my class never fail to state on their year-end evaluations that I spend far too much time on proofs (read: hand-waving plausibility arguments) and not enough time on examples (read: model problems to be followed mechanically in doing the homework).

In the sophomore year we try to make amends for this lack of rigor by taking our majors aside and teaching them calculus a second time. This time the theoretical aspects of calculus are highlighted; here we do the theory up "right" with epsilon's and delta's, definitions, theorems, and proofs. The course content is drawn primarily from differential calculus; we venture into integral calculus only to prove the Fundamental Theorem of Calculus, and sequences and series barely get mentioned. Now, if you add up all the time needed to teach this much theory, you come up about half a semester short.

Rather than beef up the course and infringe upon the content of the upper level real analysis course, we decided upon a more imaginative solution: why not spend the first half of the course learning some elementary logic and mathematical proof strategies? Since students invariably have trouble when they hit courses that require proof making skills, why not prime them a bit by giving them a peek at the various proof techniques mathematicians carry around in their toolkits and letting them try these techniques out on some familiar mathematics, such as the calculus they already know and some other, more elementary mathematics?

Such a course presents a pretty problem for the instructor, however. What text should be used for such a hybrid? Well, the introductory calculus text supplemented by class notes should suffice for the second half of the course, but what about the first half on logic and mathematical proof? It just so happened that there was a booklet at hand which addressed precisely this need: Marvin Bittinger's LOGIC, PROOF, AND SETS (1970; 1982). This was used for a couple of years at Dordt without too much complaint, but when I began teaching the course in 1983, I decided to try out the recent book by Daniel Solow which had been promoted at the annual AMS / MAA meetings, HOW TO READ AND DO PROOFS (1982).

Unfortunately, for all the merits of the book, Solow still leaves a number of things out of his discussion of mathematical proof, and some of what he does have is a bit confused according to a logician's way of thinking. So it turned out that this part of the course also needed instructor input by way of supplemental notes and corrective remarks. Since I also teach an introductory logic course for the philosophy department, it was natural for me to try to incorporate some things from that course into my course on mathematical proof. One of the books I was using there to orient myself to philosopher's logic (never having taken such a course) was THE LOGIC BOOK (1980) by Bergmann, Moor, and Nelson. This text stresses a natural deduction system and a deduction schema for formal argumentation which derives ultimately from the ideas of Gentzen and Jaŝkowski in the mid 1930's but more immediately from the 1952 textbook, SYMBOLIC LOGIC: AN INTRODUCTION, written by F. B. Fitch. Coming out of a background in mathematical logic, where a deduction system is set up almost solely to do metamathematics, I found their presentation both interesting and novel. As I mulled over the various natural deduction rules of inference, I became keenly aware of how nicely they paralleled the way mathematicians actually argue. And I became less happy with both Solow and Bittinger for teaching proof.

Having a better idea of what I wanted to do in the course, I began to scout around for another textbook, but with no success. There didn't seem to be any other books in print which aimed to teach mathematics students the art of proof making. There were lots of philosopher's logic texts which dealt with deductive argumentation, but they naturally didn't stray into mathematics. And there were a number of mathematical logic texts, but they were designed primarily with foundational issues in mind and so usually didn't discuss mathematical argumentation, only metalogical and metamathematical results.

In a moment of frustration and weakness, I decided to write my own textbook for the course, but I wondered why I should have to. Why were there so few booklets for what I wanted? Presumably because Dordt's way of making the transition to more advanced mathematics was unique. No market, no product. But why wasn't there a market for such a book? Mathematics is admittedly much more than proof and formalism, but every student of mathematics has to learn how to read and write proofs. Do mathematical educators think that logic is irrelevant to accomplishing this task? Really?? If so, why? There must be a good story behind all this, I thought. What is it? Naturally, this called for some historical journalism.

In good journalistic fashion, I will be more tentative and tantalizing than definitive and detailed. A fussier, more substantial paper would take longer to develop and give, and at any rate does not presently exist. As your reporter, I will trace the story back to some possible sources, but I will not attempt to nail the culprits with incontrovertible evidence. Nevertheless, I will not pretend to be completely neutral. As you will detect from my account, I think the mathematical community's track record on this issue is less than laudatory.

I will begin by briefly discussing a contemporary negative response to the thesis that logic ought to be used to teach students how to read and write mathematical proofs. Following this, I will survey the way in which mathematics and logic have gotten along in the past. I will then make some summary remarks about what nineteenth and twentieth century mathematicians have come to expect (and not to expect) from logic. Finally, I will draw some educational conclusions from the narrative and explain what I think can be done to go beyond our present practice in addressing the problem of learning how to do proof.

A FIRST ROUND ON THE ISSUE

A number of practicing mathematicians over the past twenty five years have been turned off by the trend in education to formalize and "rigorize" all of mathematics. They believe that the push for more rigor has been overdone, even on the undergraduate level. For them, concern over rigorous argument and axiomatic foundations is more like nitpicking than doing real mathematics. They admit that mathematics involves deductive reasoning, but argue that this is not its essence. Moreover, they believe that this aspect of mathematics is best learned the way a medieval apprentice learned his trade: by watching and doing. One need only stir the quiescent coals of the students' innate power to reason and fan them to a hot flame by providing proper models to be emulated and giving gentle correction to their first attempts. People knew how to reason before Aristotle and Frege! Archimedes and Newton required no detailed study of logic before they were able to demonstrate their mathematical results.

Paul Halmos puts the argument rather nicely. Let me quote from an article he wrote a little over a decade ago in the January 1977 issue of MATHEMATICS MAGAZINE, entitled "Logic from A to G" [i.e. Aristotle to Gödel]:

> Originally "logic" meant the same as "the laws of thought" and logicians studied the subject in the hope that they could teach these arts to all mankind. Experience has shown, however, that this is a wild-goose chase. A normal healthy human being has

> built in him all the "laws of thought" anybody has ever invented, and there is nothing that logicians can teach him about thinking and avoiding error. This is not to say that he knows how he thinks and it is not to say that he never makes errors. The situation is analogous to the walking equipment all normal healthy human beings are born with. I don't know how I walk, but I do it. Sometimes I stumble. The laws of walking might be of interest to physiologists and physicists; all I want to do is to keep on walking.

Halmos is not denigrating proof and reasoning here, though he would certainly want to stress that mathematics is much more than that. He just thinks that logic has almost nothing to offer the mathematics student, even on this limited front. Mathematical reasoning is an art, not a science; one cannot prescribe set rules for its application.

By way of a personal response, let me first substitute a fairer analogy for Halmos's walking-being. Take someone who wishes to become an accomplished painter. It is quite possible that he or she can become an artist without ever receiving formal training in perspective, color, shading, composition, various media, etc. Yet native abilities in these aspects of the trade can be corrected and sharpened by means of conscious study, and art schools subject their students to just such a training. I believe that mathematics departments should provide a similar service for their majors. In claiming that mathematical educators have a responsibility to help their students learn how to construct valid arguments, I wish to say no more than that they must awaken and sharpen abilities which are already present. This must naturally be done by means of much practice and correction - nobody disputes that - but I think it should be assisted by a close study of the common proof strategies used in mathematical reasoning.

A question which is still in order at this point concerns the means of such a study. Is logic the appropriate foundation on which to build such an art? I assume that it is, but why doesn't everyone agree? What makes mathematicians like Halmos and others turn away from logic? I'd like to offer a partial answer from the history of mathematics. So we will now turn to trace some of the history of the inter-relationships between logic and mathematics.

RELATION BETWEEN LOGIC AND MATHEMATICS IN HISTORICAL PERSPECTIVE

Mathematics, even axiomatically organized mathematics, existed before any system of logic was invented. In fact, Aristotle, who is the undisputed father of logic, seems to have drawn upon mathematics for inspiration and ideas as he developed his system of syllogistic reasoning and his theory that true science should be organized axiomatically. In return, his compatriot Euclid of a generation later seems to have adhered to Aristotle's view concerning how a demonstrative science like geometry should be structured, though he did not attempt to cast his proofs in the form of linked syllogisms. Notwithstanding this lack of orthodoxy, for over two thousand years mathematics, and particularly Euclidean geometry, was taken to be the primary prototype of "logical" reasoning, for it was thought that if its arguments were laid out in picky detail they would become strings of syllogisms. Up until the nineteenth century, then, Aristotle's system of logic was generally acknowledged to be the underlying logic for all deductive reasoning, mathematical or otherwise, though it might need to be expanded slightly to accommodate a few forms of reasoning that Aristotle had overlooked.

Scientists and mathematicians of the early modern era, however, turned their backs upon deductive logic, deeming it sterile and scholastic, unproductive of new truths. Their interest was in generating new discoveries, and this was done by means of experimentation and symbolic calculations, not by disputation or deductive argumentation, and certainly not by chains of syllogistic arguments. There was thus a conscious rejection of traditional logic by the leading thinkers of the age.

In mathematics, the chasm widened between logic and mathematical practice. It is well known that mathematics became less logically cohesive with the genesis and development of analysis in the seventeenth and eighteenth centuries. The practice of mathematics belied its logical reputation to such an extent that the ideas and techniques of the calculus were criticized (with a large measure of justification) by Berkeley and others as vague and contradictory.

It might be suspected that mathematics and logic would be brought together again once the foundations of mathematics were shored up and made more rigorous, but this is not quite the way it happened. The arithmetization of analysis accomplished by Cauchy, Weierstrass, Cantor, and Dedekind provided a firm theoretical foundation of mathematical concepts on which to erect the calculus, but the logical structure of analysis went largely unexamined, and the fields of logic and mathematics were not really drawn closer together by all this work. Logic did play an important role in the axiomatization of geometry effected by Pasch, Hilbert, and others around the end of the nineteenth century, but by this time mathematics and logic had already begun rapproachment along another path, one involving, of all fields, elementary algebra, which lacked both logical rigor and a proper foundation.

It was in England about 1850, arising out of the work of Augustus De Morgan and George Boole, that a tighter link between logic and mathematics began to be forged. Prior to this time, logic and mathematics had been kept apart in Great Britain, though lip service was still paid to mathematics as the great model of logical reasoning. As a matter of fact, the two great English universities had more or less divided up mathematics and logic between them and gone their separate ways. For while Oxford and Cambridge were both dedicated to teaching Christian gentlemen to think and converse rationally, they differed widely over how this was to be accomplished. At Oxford students were taught Aristotelian logic and were required to dispute various theses pro and con in strict syllogistic form, while in Cambridge logic and disputation had given way to an increased emphasis upon the study of mathematics as the model of exact reasoning. Contemporary attitudes toward the value of logic for learning how to do mathematical proofs find definite precedents in this split between Oxford and Cambridge.

De Morgan, though trained at Cambridge, abandoned its narrow outlook. He held that mathematics students could profit from a knowledge of the various kinds of valid arguments, so he urged fellow mathematicians to acquaint themselves with logic, which at that time was still a variant of Aristotelian logic. His writings repeatedly espoused this theme, and he attempted to develop a new and expanded system of logic.

George Boole, a self-taught mathematician and a friend of De Morgan, was stimulated by certain events (which we cannot recount here) to make a fresh investigation of logic for himself. Boole's genius was to use algebraic or symbolic computations to calculate the consequences of a set of given statements. Although he was not aware of it, his work provided partial fulfillment of Leibniz's dream a century and a half earlier about inventing an instrument by means of which two people could sit down to calculate the truth of any matter.

Boolean logic brought mathematics and logic into very close contact, but the way in which the two fields were connected kept numerous philosophers and traditional logicians at arm's length. The main stumbling block for them was the artificiality and partial uninterpretability of Boole's mathematical method of generating conclusions from premises. In keeping with the spirit of his time, Boole claimed that his system of logic had uncovered the true laws of thought, but its deductive apparatus failed to parallel the way people ordinarily reason, in mathematics or daily discourse. For this reason, many viewed his system more as a strange system of algebra than as a serious competitor to Aristotelian logic. Nevertheless, Boole did attract a number of mathematically minded followers, and through his work and that of his successors Jevons, Peirce, and Schröder, who incorporated a number of ideas from De Morgan's system as well, mathematical logic was successfully born. Before the time of Boole and De Morgan, logic had been the sole domain of academic philosophers; after their work, mathematicians increasingly claimed logic as their own, as a branch of mathematics worthy of study, much like geometry, algebra, or analysis.

In the last quarter of the nineteenth century Gottlob Frege proposed another kind of connection between mathematics and logic. Frege intended to define all the primitive concepts of mathematics in terms of logical concepts and then prove the first principles of mathematics as theorems of logic. Logic had a dual role to play in this logicist program of foundations: it was both content (conceptual foundation) and method (rigorous derivation). Frege's approach to relating mathematics and logic was novel in both regards. In order to guarantee fully rigorous derivations, Frege restricted himself to certain rules of inference, expressly stated at the outset. In order to show that mathematics was merely complex logic, Frege was also obliged to provide an adequate foundation of logical laws from which to derive mathematical propositions. It is this last aspect, logic as conceptual and theoretical basis, which seems to have received the most press in Frege's logicist philosophy of mathematics, but the other was also very important. Before Frege's work, the goal of logical rigor or deductive completeness was more rhetoric than reality in mathematics; after it, it became a real possibility.

Notwithstanding this connection between logic and proof, Frege's work did not analyze mathematical demonstration as it is practiced by mathematicians. In fact, given his outlook that inference must always proceed from true premises, Frege tended to slight conditional and indirect proofs, two of the most important strategies of mathematical argumentation. Moreover, in the process of reacting to psychologism in logic, Frege showed an unconcern for having logic's deduction system mirror the way in which valid arguments actually proceed, in mathematics or elsewhere. Frege was adamant that logic studies logical relationships among true propositions, not the laws of thought, conceived of as the rules governing the way people ordinarily reason. While logic was taken to be the underlying foundation for mathematics, then, it was not expected to furnish the underlying logic for mathematical argumentation.

Early in the twentieth century Bertrand Russell discovered that Frege's system entailed a crucial contradiction. Learning of this flaw, Frege attempted to root it out, but without success. He finally abandoned his approach altogether. Russell, on the other hand, retained his faith in the logicist program. Working with Alfred North Whitehead, Russell modified and extended it in their three volume work, PRINCIPIA MATHEMATICA. Using Peano's superior logical notation, they were able to bring it to the attention of many mathematicians and logicians. They developed their system in elaborate detail, going quite beyond Frege's work in an attempt to substantiate their approach. On the matter of coordinating logic with ordinary mathematical reasoning, however, Russell made no advance over Frege.

In the meantime two other approaches to the foundations of mathematics had begun to flourish. They also had something to say about the proper relationship between logic and mathematics. The intuitionist philosophy of mathematics developed by Brouwer denied that logic had much of a role to play at all. It certainly did not provide the theoretical foundation of mathematics, but it also could not capture the mathematical process. Mathematics was a creative enterprise which could not be contained within the bounds of logic.

David Hilbert, on the other hand, sought to clarify and extend the role of logic with respect to mathematics. Whether or not logic was taken to be the proper conceptual foundation for mathematics was in some sense irrelevant to the formalist program of mathematics which he was developing. Regardless of its origin or intended meaning, the axiomatic foundation of a mathematical theory should be considered to be a set of uninterpreted sentences, a series of symbols combined in certain allowable ways and adopted as the axioms of that system. Logic would enter this stage of the formalist program in three ways: by explicitly stipulating the ways in which symbols could be combined (the syntax); by contributing certain logical truths or tautologies as axioms to be used within the system; and by providing the rules of inference by means of which theorems could be proved (the deduction system). To this point Hilbert merely adopted what had been done by the logicists.

Once a theory was set up, however, logic would be given a completely new role to play, one that had only been hinted at before. It was Hilbert's intent to investigate the metamathematical properties of a theory. The key prerequisite of all theories was that they be logically consistent, but Hilbert was also concerned with the notions of independence and completeness. In order to show that a theory had or did not have such properties, Hilbert and others developed proof theory and the method of interpreting sentences in terms of models, both of which came to be considered integral parts of mathematical logic. Logic thus not only had the old internal task of logically unfolding a deductive theory, but the new external task of demonstrating that the theory as given had various metamathematical properties.

The work of Gödel and other mathematical logicians since 1930 has been largely along the new lines of investigation opened up by Russell and by Hilbert. With their work, mathematical logic became a tool primarily for analyzing the logical structure and capabilities of different systems of logic and axiomatic theories. Mathematical logic became synonymous with foundational investigations in logic and mathematics. Consequently, the internal role of logic in developing a deductive theory became further devalued, even molded to better suit the external role. Whether or not the rules of inference of a given deduction system were appropriate for developing the theory was considered largely irrelevant, since formal systems were now designed exclusively to facilitate foundational investigations. This led to what are now termed "Hilbert-style" deduction systems (though they can be traced back to Frege and Russell), in which a number of tautologies are adopted as axioms or laws of logic but relatively few rules of inference. Suppositional rules of inference, which posit temporary suppositions to be discharged before the end of the proof, make metalogical concerns such as the soundness of the deductive system more complicated and thus are not usually adopted in foundational investigations, though they are certainly used on the metamathematical level (who reasons without them?) and even appear in the guise of a metamathematical result, such as the deduction theorem.

EXPECTATIONS MATHEMATICIANS HAVE / DO NOT HAVE OF LOGIC

Given these developments in logic during the nineteenth and twentieth centuries, it should be apparent that the time honored role of traditional logic as providing the deep structure of deductive argumentation was gradually eroded. It must be admitted that mathematics never did exhibit the strict logical structure dictated to it by the logicians, so perhaps mathematicians were simply used to thinking of rigor as being no more than irrelevant rhetoric. Ironically enough, however, just as logic became more complex and better able to fulfill this aim, it also grew away from it.

With Boolean logic we have a system for generating consequences, but one which does not mirror the way in which arguments are actually made. Logicism gave logic a more glamorous role to play as the theoretical foundation of mathematics, and so turned attention away from the more pedestrian role it had previously been thought to have. Derivations and rules of inference were important, but only as a means to an end. Whether or not the deductions were artificial was not really relevant. Moreover, the deductions that were made were so highly symbolic and detailed and proceeded as such a snail's pace that they didn't look anything like the discourse mathematicians called "reasoning". Finally, the exciting possiblities opened up for logic by the formalist program proved so attractive and fertile that mathematicians and logicians were bewitched into thinking that the only genuine way in which to relate logic and mathematics was by way of foundational investigations. In each case the original aim of logic was laid aside.

Thus, while nineteenth and twentieth century developments drew mathematics and logic increasingly closer together and provided a great boost to the development of mathematical logic, even making this period a golden era in the history of logic, it also gave a definite direction to the development of logic and restricted the scope of the inter-connection between mathematics and logic. Since the impulse to investigate logic now came almost completely out of foundational research, logic was seen as relevant primarily for investigating metatheoretical issues. Its everyday relevance for doing mathematics, for analyzing and formulating the underlying logic of mathematical proof, was overshadowed and subverted by grander philosophical and foundational programs.

Given this background, it is no wonder that few textbooks attempt to use logic to train mathematics students in the art of proof making. Logic might seem to be the natural choice, but mathematicians have not been adequately shown the relevance and value of logic for this task. Mathematical logicians have had more important things on their agendas, and so the mathematical community has been left without their expert input on this matter. This is so in spite of the fact that there are a number of good textbooks in mathematical logic which one can draw upon. For, since these books are slanted toward metalogical and metamathematical concerns, educators who look to them for guidance on this matter find little there, and what they do find are often the wrong things. The deduction systems in such books are embryonic and unnatural, hardly what is needed for teaching students how to do proofs. Such essential proof techniques as conditional proof, proof by contradiction, proof by cases, and so on, which are the bread and butter of ordinary mathematical argumentation, are generally overlooked in favor of rules which are easier to treat from the standpoint of metalogic. What is needed in order to teach students how to write valid proofs is an emphasis upon logic as the underlying theory of mathematical argumentation, not the logicist emphasis upon logic as the content of the foundation of mathematics, nor the formalist emphasis upon logic as the tool for investigating its foundations. Natural deduction systems are called for which pay less attention to metatheory and more attention to ordinary mathematical argumentation. Tautologies and truth tables should be moved off center stage, and they should not be construed as the basis for admitting a rule of inference. An appropriate deduction system should be placed in the spotlight, so that students can learn the rich variety of proof strategies available to them.

It is my belief that this is the correct route to take in teaching mathematical proof construction. Leaving students to pick up the art of proof making by osmosis signals our failure as teachers of mathematics. We ought to address this issue instead in the spirit of De Morgan, using logic to learn the tools of proof making. We now have the logical equipment he lacked: a wonderful analysis of sentence structure, deriving from the work of Boole and Frege on sentential and predicate logic; and a natural deduction system, deriving in part from Gentzen's work in proof theory but more importantly from the work of Jaŝkowski on natural deduction, both dating from around 1934 (going into these developments would require another talk, which I'm not yet prepared to give). Natural deduction systems have existed now for around 50 years; they have been incorporated in a number of philosophical logic textbooks for over 30 years. It's high time the mathematical community familiarizes itself with them and makes use of them to construct a textbook teaching students how to read and write mathematical proofs.

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